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Symmetry classes of variable coefficient nonlinear Schrödinger equations

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Abstract. A variable-coefficient nonlinear Schrödinger (VCNLS) equation, involving three arbitrary complex functions of space-time (in 1 + 1 dimensions) is analysed from the point of view of its symmetries. All equations of the type studied having non-trivial Lie point symmetry groups G are identified. The symmetry group is shown to be at most five-dimensional and only when the equation is equivalent to the NLS equation itself or to a rather special complex Ginzburg–Landau equation. Lie point transformations are used to obtain solutions of specific VCNLS equations that should be of interest in nonlinear optics or other branches of physics.

1. Introduction

The present article is devoted to a study of variable-coefficient nonlinear Schrödinger (VCNLS) equations

$$\begin{aligned} &\mathrm{i}\,u_t + f(x,t)\,u_{xx} + g(x,t)\,u\,|u|^2 + h(x,t)\,u = 0 \\ &f = f_1 + \mathrm{i}\,f_2 \qquad g = g_1 + \mathrm{i}\,g_2 \qquad h = h_1 + \mathrm{i}\,h_2 \\ &f_j, g_j, h_j \in \mathbb{R} \quad j = 1, 2 \qquad f_1 \neq 0 \qquad g_1 \neq 0\,. \end{aligned} \tag{1.1}$$

We shall classify such equations into equivalence classes under the transformations

$$\begin{aligned} &u(x,t) = U(\tilde{x}, \tilde{t}, \tilde{u}(\tilde{x}, \tilde{t})) \qquad \tilde{x} = X(x,t) \qquad \tilde{t} = T(x,t) \\ &\frac{\partial U}{\partial \tilde{u}} \neq 0 \qquad \begin{vmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial t \\ \partial\tilde{t}/\partial x & \partial\tilde{t}/\partial t \end{vmatrix} \neq 0\,. \end{aligned} \tag{1.2}$$

Each equivalence class will be characterized by its Lie point symmetry group G. We shall see that the symmetry group is at most five-dimensional. The existence of non-trivial symmetry groups of course imposes strong restrictions on the complex functions f_j, g_j and h_j in (1.1). The symmetries will then be used to obtain physically interesting solutions with a quite non-trivial space-time dependence.

We shall call transformations of the form (1.2), leaving the form of the VCNLS equation invariant, but possibly changing the functions f, g and h into different ones ('allowed transformations'). The classification method used here has recently been applied to the study of the variable-coefficient Korteweg–de Vries equation in a similar manner [1, 2].

The motivation for the present study lies in the physical importance of the VCNLS equation. Equation (1.1) is a natural extension of two fundamental equations. One is the nonlinear Schrödinger (NLS) equation itself, obtained from (1.1), for $f_1 = 1$, $g_1 = \pm 1$ and $f_2 = g_2 = h_1 = h_2 = 0$. The complex function $u(x,t)$ has different physical meanings in different branches of physics. It may be an electromagnetic potential and the NLS equation then describes, for instance, the evolution of nonlinear Langmuir waves in a plasma [3]. In other applications, $u(x,t)$ may be a wave amplitude, and the NLS equation describes weakly nonlinear, weakly dispersive waves in fibre optics [4] or deep water [5]. Independent of its physical interest, the NLS equation is a prototype soliton equation, having all the attributes of an infinite-dimensional completely integrable Hamiltonian system [6–8]. Large families of solutions of the NLS equation are obtained by essentially linear techniques (the inverse scattering transform and its generalizations). Such solutions include solitons, multisolitons, breathers and quasiperiodic solutions.

Equation (1.1) also generalizes another fundamental equation, the complex Ginzburg–Landau (CGL) equation, obtained when f, g and h are constant (the real Ginzburg–Landau equation is obtained when they are purely imaginary). In this case $u(x,t)$ can be a complex order parameter, describing various physical phenomena close to critical stability. In hydrodynamics, for instance, it results from an expansion in some parameter (e.g. the Reynolds, Rayleigh or Taylor number) near the critical value of that parameter. For example, it is the generic amplitude equation that governs the initial stages of phase transition in plane Poiseuille flow [9] (fluid flowing between two parallel plates), Taylor–Couette flow [10] (fluid flowing between two rotating cylinders) as well as Rayleigh–Bénard convection [11] (fluid with a vertical temperature gradient). Similarly to fluid systems, the CGL equation is also found to govern the appearance of chemical turbulence in reaction-diffusion systems [12].

In addition to critical phenomena, the CGL equation also has numerous applications in the modelling of the electric field amplitude in nonlinear optics. For instance, it describes, under appropriate conditions, the dynamics of light in laser cavities [13–17] and semiconductors [18]. Recently, it has also been used to model the dynamics of a spatial solitary wave in a saturated amplifying/absorbing medium [19] and the dynamics of pulse propagation in nonlinear rare-earth doped optical fibres for which material dispersion, gain dispersion and nonlinearity all contribute significantly [20–26].

Because of its wide range of applications, properties of the CGL equation are continuously the subject of studies both in physical and mathematical contexts. Among the properties already known, let us mention the following ones. The Hirota method [27] has been used to rewrite the CGL equation in a bilinear form in order to obtain exact solutions describing solitary waves and shock fronts [28]. Numerical integrations of the CGL equation were also performed and led to the determination of coherent structures with complex field profiles [29]. A stability criterion was obtained which determines whether the system underlying the CGL equation does or does not evolve into a monochromatic state [30]. In relation to this criterion, the bifurcation structure and asymptotic dynamics of unstable periodic modulations of a uniform wave-train were also studied [31–37]. In particular, asymptotic states such as limit cycles, 2-torus and chaotic attractors were shown to exist. Many investigations of periodic solutions have been performed [38–40]. Finally, as far as we know, only one study was concerned with the question of symmetry properties of the CGL equation [41].

Allowing the parameters f, g and h to be complex functions of the independent variables x and t may correspond to new or more realistic physical conditions. In hydrodynamics, for instance, variable depth, and the presence of vorticity or viscosity are a few examples of physical effects that introduce such variable coefficients in the KdV equation [42]. In nonlinear optics, non-homogeneous dielectric media usually lead to variable coefficients in the NLS or CGL equation. Propagation-coordinate-dependent amplification or transverse-coordinate-dependent diffraction and nonlinearity may also be modelled in this way.

The generalization of the NLS to the VCNLS equation usually destroys some or all of the integrability properties of the original equation. There exist, however, particular cases when this is not so [43].

Finally, we mention that other generalizations of the NLS equation have been studied in the literature. Usually they involve constant coefficients, but generalize the type of nonlinearity [44–50].

The aim of the present study is two-fold. First, in sections 2 and 3, we will classify the VCNLS equations of the form (1.1) according to the dimension and type of their point-symmetry groups, i.e. the set of Lie point transformations of type (1.2) that preserve the form of the equation and transform solutions amongst each other. The classification method is based on the usual infinitesimal techniques for finding point symmetry groups of differential equations [50, 51] and will be described in section 3. The representative equation in each class will be determined by an extensive use of the concept of 'allowed transformations', which will be presented in section 2. These transformations are those that relate equations of the form (1.1) to other equations of the same form, but possibly with different arbitrary functions f, g and h. Lie point symmetry transformations are particular cases of allowed transformations when the form of the functions f, g and h is preserved. Two equations related by an allowed transformation will be considered to belong to the same equivalence class. As a result, the symmetry group of the VCNLS equation will be shown to be at most five-dimensional, which occurs only if the function f can be transformed into a real constant, g to a complex constant and h to 0.

Second, in section 4, we will concentrate on the analysis of a physically important subset of allowed transformations, i.e. those that relate the VCNLS equation to the CGL and NLS equations. In particular, our analysis will permit the identification of the form of the VCNLS equation that possesses the same symmetry and integrability properties as the NLS equation.

2. Allowed transformations

2.1. General form of the transformations

Let us now determine the Lie point transformations that leave the VCNLS equation (1.1) form invariant. Such transformations, by definition, do not add any terms to the considered equation, but may change the functions that are already there. We restrict ourselves to fibre-preserving transformations, i.e. we assume that they have the form of (1.2). In other words, the new independent variables $\tilde{t}$ and $\tilde{x}$ do not depend on u.

We calculate u_t and u_{xx} and substitute into (1.1). Requiring that the equation for $\tilde{u}(\tilde{x}, \tilde{t})$ be linear in $\tilde{u}_{\tilde{x}\tilde{x}}$ and $\tilde{u}_{\tilde{t}}$ and that no terms of the type $\tilde{u}_{\tilde{t}\tilde{x}}$ occur, we find

$$U_{\tilde{u}\tilde{u}} = 0 \qquad \tilde{t}_x = 0 . \tag{2.1}$$

Requiring that the nonlinearity be cubic (no quadratic terms in u present), and that terms proportional to $\tilde{u}_{\tilde{x}}$ cancel, as well as terms not involving the dependent function at all, we obtain the allowed transformations

$$u(x, t) = Q(x, t)\, \tilde{u}(\tilde{x}, \tilde{t}) \qquad \tilde{x} = X(x, t) \qquad \tilde{t} = T(t) . \tag{2.2}$$

We can view $T(t)$ and $X(x,t)$ as arbitrary real sufficiently smooth functions. The function $Q(x,t)$ is complex and must satisfy

$$\mathrm{i}\, Q\, X_t + f(x,t)\ (X_{xx}\, Q + 2\, Q_x\, X_x) = 0\,. \tag{2.3}$$

The transformed functions in the VCNLS equation are

$$\begin{aligned} \tilde{f}(\tilde{x},\tilde{t}) &= f(x,t)\frac{X_x^2}{\dot{T}} \qquad \tilde{g}(\tilde{x},\tilde{t}) = g(x,t)\frac{|Q|^2}{\dot{T}} \\ \tilde{h}(\tilde{x},\tilde{t}) &= \frac{1}{Q\dot{T}}\{h\, Q + \mathrm{i}\, Q_t + f\, Q_{xx}\} \end{aligned} \tag{2.4}$$

where x and t must be expressed in terms of $\tilde{x}$ and $\tilde{t}$ from (2.2) (the dot stands for a time derivative).

2.2. *Simplification of the equation and restrictions on allowed transformations*

Since we shall always assume $f_1(x,t) \neq 0$ in some open interval of $\mathbb{R}$ we can set $X_x^2 = \pm\dot{T}\, f_1^{-1}$. This amounts to putting

$$f(x,t) = 1 + \mathrm{i}\, f_2(x,t) \qquad f_2 \in \mathbb{R} \tag{2.5}$$

in (1.1).

In order to keep this normalization intact, we must limit the allowed transformations to

$$\begin{aligned} &u(x,t) = Q(x,t)\,\tilde{u}(\tilde{x},\tilde{t}) \qquad \tilde{t} = T(t) \qquad \tilde{x} = \sqrt{\dot{T}(t)}\, x + \xi(t) \\ &Q \in \mathbb{C} \qquad T,\xi \in \mathbb{R} \qquad \dot{T} > 0 \end{aligned} \tag{2.6}$$

with

$$\mathrm{i}\, Q\left[\left(\sqrt{\dot{T}}\right)^{\cdot} x + \dot{\xi}\right] + 2(1+\mathrm{i}\, f_2)\, Q_x \sqrt{\dot{T}} = 0\,. \tag{2.7}$$

We introduce the moduli and phases of u and Q, putting

$$\begin{aligned} &u(x,t) = \rho(x,t)\,\mathrm{e}^{\mathrm{i}\omega(x,t)} \qquad Q(x,t) = R(x,t)\,\mathrm{e}^{\mathrm{i}\phi(x,t)} \\ &\rho \geqslant 0 \qquad R \geqslant 0 \qquad 0 \leqslant \omega < 2\pi \qquad 0 \leqslant \phi < 2\pi\,. \end{aligned} \tag{2.8}$$

Equation (2.7) then implies

$$\frac{R_x}{R} = -\frac{f_2}{2(1+f_2^2)}\frac{\left(\sqrt{\dot{T}}\right)^{\cdot} x + \dot{\xi}}{\sqrt{\dot{T}}} \tag{2.9}$$

$$\phi_x = -\frac{1}{2(1+f_2^2)}\frac{\left(\sqrt{\dot{T}}\right)^{\cdot} x + \dot{\xi}}{\sqrt{\dot{T}}}\,. \tag{2.10}$$

These allowed transformations change the functions $f(x,t)$, $g(x,t)$ and $h(x,t)$ in the VCNLS equation as follows:

$$\tilde{f}(\tilde{x},\tilde{t}) = 1 + \mathrm{i}\, f_2(x,t) \qquad \tilde{g}(\tilde{x},\tilde{t}) = g(x,t)\frac{R(x,t)^2}{\dot{T}(t)} \tag{2.11a}$$

$$\begin{aligned} \tilde{h}(\tilde{x},\tilde{t}) = \frac{1}{\dot{T}}\Bigg\{ & h_1(x,t) - \phi_t + \frac{R_{xx} - R\phi_x^2}{R} - f_2\frac{2\, R_x\,\phi_x + R\,\phi_{xx}}{R} \\ & + \mathrm{i}\left[h_2(x,t) + \frac{R_t}{R} + f_2\,\frac{R_{xx} - R\,\phi_x^2}{R} + \frac{2\, R_x\,\phi_x + R\,\phi_{xx}}{R}\right]\Bigg\}. \end{aligned} \tag{2.11b}$$

2.3. Transformations of vector fields

We shall see in section 3 that the Lie algebra of the symmetry group of (1.1) is realized by differential operators of the form

$$X = \tau(t)\,\partial_t + \left[\tfrac{1}{2}\,\dot{\tau}x + \alpha(t)\right]\partial_x + A(x,t)\,\rho\,\partial_\rho + D(x,t)\,\partial_\omega \tag{2.12}$$

where the real functions $\tau(t)$, $\alpha(t)$, $A(x,t)$ and $D(x,t)$ are subject to further determining equations. Under the allowed transformations (2.6) the vector field X transforms into a vector field of the same form. If we have $\tau(t) \neq 0$ in (2.12), we can put $\dot{T} = \pm\tau^{-1}$ and $\dot{\xi} = -\alpha\sqrt{\dot{T}}\tau^{-1}$ to obtain

$$X_1 = \partial_t + A(x,t)\,\rho\,\partial_\rho + D(x,t)\,\partial_\omega \tag{2.13}$$

(A and D are not the same as in (2.12)).

Moreover the form of X_1 in (2.13) is not changed by allowed transformations (2.6) with $\dot{T} = 1$, $\dot{\xi} = 0$ and $R = R(t)$, $\phi = \phi(t)$. Such transformations will amount to the substitution

$$A(x,t) \to A(x,t) - \frac{\dot{R}(t)}{R} \qquad D(x,t) \to D(x,t) - \dot{\phi}(t) \tag{2.14}$$

in (2.13).

For $\tau = 0$, $\alpha \neq 0$ in (2.12), we can transform the vector field into

$$X_2 = \partial_x + A(x,t)\,\rho\,\partial_\rho + D(x,t)\,\partial_\omega\,. \tag{2.15}$$

Further allowed transformations with $\dot{T} = 1$ change the functions A and D in (2.15) into

$$A \to A + \frac{f_2\,\dot{\xi}}{2(1+f_2^2)} \qquad D \to D + \frac{\dot{\xi}}{2(1+f_2^2)} \qquad t \to t \qquad x \to x + \xi(t)\,. \tag{2.16}$$

3. The symmetry group

3.1. The determining equations

The Lie algebra L of the symmetry group of (1.1) will be realized by vector fields

$$X = \eta_1\,\partial_x + \eta_2\,\partial_t + \phi_1\,\partial_\rho + \phi_2\,\partial_\omega \tag{3.1}$$

where η_1, η_2, ϕ_1 and ϕ_2 are functions of x, t, ρ and ω (ρ and ω are the modulus and phase of the function u). The algorithm for determining these functions is described, for example, in [51, 52]. We use a MATHEMATICA version [53] of the MACSYMA program [54] which calculates the second prolongation, $\mathrm{pr}^{(2)}\,X$, applies it to (1.1) (written as a system of two real equations), and imposes that the result should vanish on the solution set of (1.1). This provides a set of 68 determining equations, i.e. linear partial differential equations for the functions η_1, η_2, ϕ_1 and ϕ_2. The code solves the simplest amongst them, uses the result to simplify the remaining ones and then prints out a system of 21 remaining equations.

Part of the determining equations can be solved independently of the functions f, g and h in (1.1). The result is that η_1, η_2, ϕ_1 and ϕ_2 satisfy

$$\eta_1 = \eta_1(x,t) \qquad \eta_2 = \eta_2(t) \qquad \phi_1 = A(x,t)\,\rho \qquad \phi_2 = D(x,t) \tag{3.2}$$

where η_1, η_2, A and D are subject to eight remaining determining equations, involving the functions figuring in (1.1).

Before spelling out these equations we recall that $f_1(x,t)$ satisfies $f_1 \neq 0$. We then use the allowed transformations to normalize $f_1(x,t) = 1$, as in (2.5). The remaining determining equations reduce to

$$\eta_2 = \tau(t) \qquad \eta_1 = \tfrac{1}{2}\,\dot\tau(t)\,x + \alpha(t) \tag{3.3}$$

$$D_x = \frac{1}{2(1+f_2^2)}\left(\tfrac{1}{2}\,\ddot\tau\,x + \dot\alpha\right) \tag{3.4a}$$

$$A_x = \frac{f_2}{2(1+f_2^2)}\left(\tfrac{1}{2}\,\ddot\tau\,x + \dot\alpha\right) \tag{3.4b}$$

$$\tau\, f_{2,t} + \left[\tfrac{1}{2}\,\dot\tau\,x + \alpha\right] f_{2,x} = 0 \tag{3.5}$$

$$\tau\, g_{1,t} + \left[\tfrac{1}{2}\dot\tau\,x + \alpha\right] g_{1,x} + (2A + \dot\tau)\, g_1 = 0 \tag{3.6}$$

$$\tau\, g_{2,t} + \left[\tfrac{1}{2}\,\dot\tau\,x + \alpha\right] g_{2,x} + (2A + \dot\tau)\, g_2 = 0 \tag{3.7}$$

$$\tau\, h_{1,t} + \left[\tfrac{1}{2}\,\dot\tau\,x + \alpha\right] h_{1,x} + \dot\tau\, h_1 - D_t + A_{xx} - f_2\, D_{xx} = 0 \tag{3.8}$$

$$\tau\, h_{2,t} + \left[\tfrac{1}{2}\,\dot\tau\,x + \alpha\right] h_{2,x} + \dot\tau\, h_2 + A_t + D_{xx} + f_2\, A_{xx} = 0\,. \tag{3.9}$$

In the following analysis we always assume

$$g_1(x,t) \neq 0 \tag{3.10}$$

i.e. the original nonlinear term in the NLS equation is present. Throughout we shall make use of the allowed transformations to simplify vector fields and the determining equations. The justification for this is that we are classifying equations of the form (1.1) according to their symmetries. The resulting VCNLS equations will represent conjugacy classes with respect to allowed transformations.

We note that the results of (3.1) and (3.3) are summed up in (2.12).

3.2. One- and two-dimensional symmetry algebras

As noted above, an element X of the symmetry algebra L will have the form (2.12). Using allowed transformations we can further simplify the considered element X. Three cases occur.

(1) $\tau(t) = 0$, $\alpha(t) = 0$
From (3.6) we obtain $A = 0$ (since we have $g_1 \neq 0$) and (3.4*a*) and (3.8) imply $D =$ constant. The result is that precisely one symmetry operator of this type exists, independently of the form of the functions f, g and h in (1.1), namely $W = \partial_\omega$. Its meaning is obvious: we can always add a constant to the phase ω of any solution $u(x,t)$. Moreover, this is the only pure gauge transformation allowed.

(2) $\tau(t) \neq 0$

We use an allowed transformation to transform X into X_1 of (2.13), i.e. set $\tau(t) = 1$, $\alpha(t) = 0$. From (3.4) we obtain $D_x = 0$, $A_x = 0$ and we use the transformation of (2.14) to set $A(t) = 0$, $D(t) = 0$. Equations (3.5)–(3.9) then imply $f_{2,t} = g_{1,t} = g_{2,t} = h_{1,t} = h_{2,t} = 0$.

(3) $\tau(t) = 0$, $\alpha(t) \neq 0$

We transform the vector field X into X_2 of (2.15), i.e. $\alpha(t) = 1$. From the determining equations we obtain that X must have the form

$$X = \partial_x - \tfrac{1}{2} q(t)\,\rho\,\partial_\rho + r(t)\,\partial_\omega \tag{3.11}$$

and that the functions in (1.1) satisfy

$$\begin{aligned} f &= 1 + \mathrm{i}\, f_2(t) \\ g &= \left[\gamma_1(t) + \mathrm{i}\,\gamma_2(t)\right] \mathrm{e}^{q(t)x} \\ h &= \left[\dot r(t)\,x + p_1(t)\right] + \mathrm{i}\left[\tfrac{1}{2}\dot q(t)\,x + p_2(t)\right]. \end{aligned} \tag{3.12}$$

For $f_2 \neq 0$ we use the allowed transformations to set $q(t) \to 0$, $p_1(t) \to 0$ and $p_2(t) \to 0$. For $f_2 = 0$ we can transform $r(t) \to 0$, $p_1(t) \to 0$ and $p_2(t) \to 0$, but $q(t)$ is an invariant.

The results obtained so far can be summed up as follows.

Theorem 1. The VCNLS equation (1.1) is invariant under gauge transformations $\tilde\omega = \omega + \omega_0$, generated by

$$S_{1,1} : W = \frac{\partial}{\partial\omega} \tag{3.13}$$

for any choice of the complex functions f, g and h. The group of pure gauge transformations (leaving x and t invariant) is not larger for any choice of f, g, h. □

Theorem 2. The VCNLS equation (1.1) has a two-dimensional symmetry algebra if and only if the functions f, g and h and the symmetry algebra can be transformed into one of the following cases:

$$\begin{aligned} &S_{2,1} : P_0 = \partial_t \qquad W = \partial_\omega \\ &f = 1 + \mathrm{i}\, f_2(x) \qquad g = g_1(x) + \mathrm{i}\, g_2(x) \qquad h = h_1(x) + \mathrm{i}\, h_2(x) \end{aligned} \tag{3.14}$$

$$\begin{aligned} &S_{2,2} : X = \partial_x + r(t)\,\partial_\omega \qquad W = \partial_\omega \\ &f = 1 + \mathrm{i}\, f_2(t) \qquad g = g_1(t) + \mathrm{i}\, g_2(t) \qquad h = \dot r(t)\,x\,. \end{aligned} \tag{3.15}$$

For $f_2(t) = 0$ we have $r(t) = 0$.

$$\begin{aligned} &S_{2,3} : X = \partial_x - \frac{q(t)}{2}\,\rho\,\partial_\rho \qquad W = \partial_\omega \\ &f = 1 \qquad g = \left[\gamma_1(t) + \mathrm{i}\,\gamma_2(t)\right] \mathrm{e}^{q(t)x} \qquad h = \mathrm{i}\,\frac{\dot q(t)}{2}\,x \qquad q \neq 0\,. \end{aligned} \tag{3.16}$$

All functions in (3.14)–(3.16) are real. □

We note that all three two-dimensional symmetry algebras $S_{2,1}$, $S_{2,2}$ and $S_{2,3}$ are Abelian.

3.3. Three-dimensional symmetry algebras

A real Lie algebra L of dimension $\dim L = 3$ can be either simple, or solvable. The simple Lie algebras $sl(2,\mathbb{R})$ and $su(2)$ cannot be realized in terms of vector fields of the form (2.12). Hence, any algebra we obtain must be solvable. All solvable Lie algebras with $\dim L = 3$ have two-dimensional Abelian ideals. We choose a basis $\{X_1, X_2, X_3\}$ with X_1 and X_2 in the ideal and write the commutation relations as

$$\begin{pmatrix} [X_1, X_3] \\ [X_2, X_3] \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad [X_1, X_2] = 0 \tag{3.17}$$

where $a,\ b,\ c,\ d\ \in \mathbb{R}$.

With no loss of generality we assume that the ideal $\{X_1, X_2\}$ is in its standard form, i.e. $S_{2,1}, S_{2,2}$ or $S_{2,3}$. We always choose $X_1 = W$ and hence have $a = b = 0$ in (3.17). For $d \neq 0$ we can set, by a change of basis, $d = 1$, $c = 0$. For $d = 0$, we have either $c = 0$ or $c = 1$. In other words the algebras we obtain can be Abelian ($a = b = c = d = 0$), nilpotent ($a = b = d = 0$, $c = 1$), or solvable and decomposable ($a = b = c = 0, d = 1$). The Abelian ideal is unique only in the solvable non-nilpotent case.

(1) Ideal $S_{2,1}$
We have $X_1 = W$, $X_2 = P_0$ and take X_3 as in (2.12). The functions f, g and h are as in (3.14). We first impose that $\{X_1,\ X_2,\ X_3\}$ forms a Lie algebra, then solve the determining equations (3.4)–(3.9) for X_3.

We obtain four distinct cases, after simplifying by allowed transformations. They are:

$$\begin{aligned} &S_{3,1} : P_0 = \partial_t \qquad P_1 = \partial_x \qquad W = \partial_\omega \\ &f = 1 + \mathrm{i} f_2 \qquad g = \epsilon + \mathrm{i}\gamma \qquad h = \mathrm{i} h_2 \end{aligned} \tag{3.18}$$

$$\begin{aligned} &S_{3,2} : P_0 = \partial_t \qquad X = \partial_x + \tfrac{1}{2}\rho\,\partial\rho \qquad W = \partial_\omega \\ &f = 1 \qquad g = (\epsilon + \mathrm{i}\gamma)\,\mathrm{e}^{-x} \qquad h = \mathrm{i} h_2 \end{aligned} \tag{3.19}$$

$$\begin{aligned} &S_{3,3} : P_0 = \partial_t \qquad X = \partial_x + a\,\rho\,\partial_\rho + t\,\partial_\omega \qquad W = \partial_\omega \\ &f = 1 + \mathrm{i} f_2 \qquad g = (\epsilon + \mathrm{i}\gamma)\,\mathrm{e}^{-2ax} \qquad h = x + \mathrm{i}b \end{aligned} \tag{3.20}$$

$$\begin{aligned} &S_{3,5} : P_0 = \partial_t \qquad D = t\,\partial_t + \frac{1}{2}x\,\partial_x - \frac{p+2}{4}\rho\,\partial_\rho \qquad W = \partial_\omega \\ &f = 1 + \mathrm{i} f_2 \qquad g = (\epsilon + \mathrm{i}\gamma)\,x^p \qquad h = (h_1 + \mathrm{i} h_2)\,\frac{1}{x^2}\,. \end{aligned} \tag{3.21}$$

The quantities f_2, γ, h_1, h_2, a and p are arbitrary real constants and $\epsilon = \pm 1$. The algebras $S_{3,1}$ and $S_{3,2}$ are Abelian, $S_{3,3}$ is nilpotent, $S_{3,5}$ is solvable and decomposable. For the algebra $S_{3,2}$ we have $f_2 = 0$; for $f_2 \neq 0$, $S_{3,2}$ would be equivalent to $S_{3,1}$.

(2) Ideal $S_{2,2}$
We take the ideal as in (3.15) and add an element X_3 of the form given in (2.12). We take f, g, and h as in (3.15) and solve the determining equations required for X_3 to be a symmetry operator. We find that in order for $\{X_1,\ X_2,\ X_3\}$ to form a Lie algebra, we must have $\tau = \tau_1 t + \tau_0$ with τ_1 and τ_0 constant. The case $\tau_1 \neq 0$ leads to a new class of symmetry algebras. The case $\tau_1 = 0$, $\tau_0 \neq 0$ gives an algebra already in the list ($S_{3,1}$ or

$S_{3,3}$). The case $\tau_1 = 0,\ \tau_0 = 0$ gives a new class of subalgebras. We represent the two new algebras by

$$\begin{aligned} S_{3,4}: P_1 &= \partial_x \qquad B = t\,\partial_x + \tfrac{1}{2}x\,\partial_\omega \qquad W = \partial_\omega \\ f &= 1 \qquad g = g_1(t) + \mathrm{i}\,g_2(t) \qquad h = 0\,. \end{aligned} \tag{3.22}$$

The function $g_1(t)$ and $g_2(t)$ are arbitrary. The operator B generates Galilei transformations

$$\begin{aligned} S_{3,6}: X &= \partial_x - \frac{2a}{\sqrt{t}}\,\partial_\omega \qquad D = t\,\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{2}\rho\,\partial_\rho \qquad W = \partial_\omega \\ f &= 1 + \mathrm{i}\,f_2 \qquad g = \epsilon + \mathrm{i}\,g_2 \qquad h = a\,\frac{x}{t^{3/2}} - \frac{\mathrm{i}\,b}{t} \end{aligned} \tag{3.23}$$

where $f_2,\ g_2,\ a,\ b$ are constants, $\epsilon = \pm 1$.

(3) Ideal $S_{2,3}$
This ideal provides one more class of Lie algebras, represented as follows:

$$\begin{aligned} S_{3,7}: X &= \partial_x - \frac{1}{2}\frac{q}{\sqrt{t}}\,\rho\,\partial_\rho \qquad D = t\,\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{2}\rho\,\partial_\rho \qquad W = \partial_\omega \\ f &= 1 \qquad g = (\epsilon + \mathrm{i}\,\gamma_2)\,\mathrm{e}^{qx/\sqrt{t}} \qquad h = -\frac{qx}{4t^{3/2}} + \frac{a}{t} \end{aligned} \tag{3.24}$$

where γ_2, q, and a are constants, $\epsilon = \pm 1$.

We again sum up the results as a theorem.

Theorem 3. Seven classes of VCCGL equations with three-dimensional symmetry groups exist. The coefficients in the equations and the Lie algebras themselves are presented in (3.18)–(3.24). The Lie algebras $S_{3,1}$ and $S_{3,2}$ are Abelian, $S_{3,3}$ and $S_{3,4}$ are nilpotent, non-Abelian and $S_{3,5}$, $S_{3,6}$ and $S_{3,7}$ are solvable and decomposable.

3.4. Four-dimensional symmetry algebras

(1) Non-solvable Lie algebras
For $\dim L = 4$ we can have a symmetry algebra of the form $sl(2,\mathbb{R}) \oplus A_1$. By allowed transformations we can take the symmetry algebra and coefficients in the equation to

$$\begin{aligned} S_{4,1}: P_0 &= \partial_t \qquad D = 2t\,\partial_t + x\,\partial_x - \tfrac{1}{2}\rho\,\partial_\rho \qquad C = t^2\,\partial_t + xt\,\partial_x - \tfrac{1}{2}t\rho\,\partial_\rho + \tfrac{1}{4}x^2\,\partial_\omega \\ W &= \partial_\omega \qquad f = 1 \qquad g = (\epsilon + \mathrm{i}\,\gamma)\,\frac{1}{x} \qquad h = (h_1 + \mathrm{i}\,h_2)\,\frac{1}{x^2} \end{aligned} \tag{3.25}$$

where γ, h_1 and h_2 are constants, $\epsilon = \pm 1$.

(2) Nilpotent Lie algebras
A nilpotent symmetry algebra will have a three-dimensional Abelian ideal. It can be transformed to the form

$$\begin{aligned} S_{4,2}: P_0 &= \partial_t \qquad P_1 = \partial_x \qquad B = t\,\partial_x + \tfrac{1}{2}x\,\partial_\omega \qquad W = \partial_\omega \\ f &= 1 \qquad g = \epsilon + \mathrm{i}\,g_2 \qquad h = \mathrm{i}\,h_2 \end{aligned} \tag{3.26}$$

where $\epsilon = \pm 1$, g_2, and h_2 are constants, $h_2 \neq 0$.

(3) Solvable non-nilpotent Lie algebras
We shall just present the results here. To obtain them we first take the nilradical (maximal nilpotent ideal) into its standard form. This is either Abelian as in $S_{3,1}$ or $S_{3,2}$, or nilpotent, as in $S_{3,3}$ or $S_{3,4}$. The algebra is then extended by adding a further element. This in turn imposes restrictions on the coefficients in the equation. The following inequivalent symmetry algebras are obtained:

$$\begin{aligned} &S_{4,3} : P_0 = \partial_t \qquad P_1 = \partial_x \qquad W = \partial_\omega \qquad D = t\,\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{2}\rho\,\partial_\rho \\ &f = 1 + \mathrm{i}\,f_2 \qquad g = \epsilon + \mathrm{i}\,g_2 \qquad h = 0 \end{aligned} \tag{3.27}$$

where $\epsilon = \pm 1$, f_2 and g_2 are constants, $f_2 \neq 0$. The algebra is decomposable as

$$\begin{aligned} &S_{4,3} = \{D, P_0, P_1\} \oplus W \qquad S_{4,4} : P_1 = \partial_x \qquad B = t\,\partial_x + \tfrac{1}{2}x\,\partial_\omega \qquad W = \partial_\omega \\ &D = t\,\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{2}\rho\,\partial_\rho \qquad f = 1 \qquad g = \epsilon + \mathrm{i}\,\gamma \qquad h = \mathrm{i}\,\frac{h_2}{t} \end{aligned} \tag{3.28}$$

where $\epsilon = \pm 1$, γ and $h_2 \neq 0$ are constants, and

$$\begin{aligned} &S_{4,5} : P_1 = \partial_x \qquad B = t\,\partial_x + \tfrac{1}{2}x\,\partial_\omega \qquad W = \partial_\omega \\ &C = (t^2+1)\,\partial_t + tx\,\partial_x - \rho\,\partial_\rho + \tfrac{1}{4}x^2\,\partial_\omega \\ &f = 1 \qquad g = \epsilon + \mathrm{i}\,\gamma \qquad h = \frac{\mathrm{i}\,2h_2 + t}{2(t^2+1)} \end{aligned} \tag{3.29}$$

where $\epsilon = \pm 1$, γ, h_2 are constants.

The algebras $S_{4,4}$ and $S_{4,5}$ are indecomposable. They are not mutually isomorphic, since D acts on P_1 and B like a Lorentz transformation, C like a rotation.

We can now sum up the results as a theorem.

Theorem 4. Five conjugacy classes of VCNLS equations with four-dimensional symmetry groups exist. The coefficients in the equation and the corresponding symmetry algebras are summed up in (3.25)–(3.29). The algebra $S_{4,1}$ has the structure of $gl(2, \mathbb{R})$, $S_{4,2}$ is nilpotent, $S_{4,3}$ is solvable and decomposable, $S_{4,4}$ and $S_{4,5}$ are solvable and indecomposable.

3.5. Five-dimensional symmetry algebras

The result in this case is very simple and we shall just present it without proof.

Theorem 5. Any VCNLS equation with a five-dimensional symmetry group can be transformed into the VCNLS equation satisfying

$$f = 1 \qquad g = \epsilon + \mathrm{i}\,g_2 \qquad h = 0\,. \tag{3.30a}$$

The symmetry algebra is solvable and has the form

$$P_0 = \partial_t \qquad P_1 = \partial_x \qquad W = \partial_\omega \qquad B = t\,\partial_x + \tfrac{1}{2}x\,\partial_\omega \qquad D = t\,\partial_t + \tfrac{1}{2}x\,\partial_x - \tfrac{1}{2}\rho\partial_\rho \tag{3.30b}$$

which is isomorphic to the one-dimensional extended Galilei similitude algebra [47].

4. Solutions of VCNLS equations

As we have seen above, the concept of allowed transformations was very useful in the classification process leading to the representative equations given in section 3. In a complementary approach, allowed transformations can be used to obtain solutions of rather general VCNLS equations in terms of solutions of a simpler representative equation that exhibits the same symmetry properties. In the present section, we briefly consider such solutions for VCNLS equations with $f_2 =$ constant, related by allowed transformations to the complex Ginzburg–Landau equation, i.e. (1.1) with f, g and h constant.

By reformulating the results of section 2, one can show that the most general VCNLS equations that can be transformed into one of the representative equations of section 3, with $f_2 =$ constant, are those for which $f_2 = \tilde{f}_2$ and

$$g = (\tilde{g}_1 + \mathrm{i}\,\tilde{g}_2)\,\dot{T}\,I^{-2}\,\mathrm{e}^{-2f_2(Kx^2+Lx)} \tag{4.1}$$

$$h_1 = [\dot{K} + 4(1+f_2^2)\,K^2]\,x^2 + [\dot{L} + 4(1+f_2^2)\,KL]\,x + \dot{J} + (1+f_2^2)\,L^2 + \tilde{h}_1\dot{T} \tag{4.2}$$

$$h_2 = -f_2[\dot{K} + 4(1+f_2^2)\,K^2]\,x^2 - f_2[\dot{L} + 4(1+f_2^2)\,KL]\,x$$
$$- \dot{I}\,I^{-1} - 2(1+f_2^2)\,K - f_2(1+f_2^2)\,L^2 + \tilde{h}_2\,\dot{T} \tag{4.3}$$

where $\tilde{}$ stands for the coefficients of the chosen representative equation and I, J, K, L and T are functions of t alone.

The transformation itself turns out to be

$$\tilde{t} = T \qquad \text{with} \qquad \dot{T} = T_0\,\mathrm{e}^{-8(1+f_2^2)\int K\,\mathrm{d}t} \tag{4.4a}$$

$$\tilde{x} = \sqrt{\dot{T}}\,x + \xi \qquad \text{with} \qquad \dot{\xi} = -2(1+f_2^2)\,\sqrt{\dot{T}}\,L \tag{4.4b}$$

$$u = \tilde{u}(\tilde{x},\tilde{t})\,I\,\mathrm{e}^{f_2(Kx^2+Lx)}\,\mathrm{e}^{\mathrm{i}(Kx^2+Lx+J)} \tag{4.4c}$$

where T_0 is a positive constant. It is important to note that the transformation (4.4) involves four arbitrary functions of t, namely I, J, K and L. This provides enough freedom to build up VCNLS equations that may correspond to a particular physical situation.

It is not our present intention to analyse the effect of the transformation (4.4) on all the representative equations with $f_2 =$ constant, since exact solutions are difficult to obtain for most of them. We shall keep to a few illustrative examples for the physically important CGL equation for which many exact solutions are known. For $f_2 \neq 0$, the functions K and L are fixed by the coefficients of x^2 and x in g respectively. For $f_2 = 0$, they are fixed by the coefficients of x^2 and x in h_1. Fixing the coefficient of x^2 in h_1 leads to a Riccati equation for K that can be solved exactly in many cases.

In the following, we choose $L = 0$ in order to conserve the mirror symmetry about $x = 0$.

Example 1. Consider first the case $f_2 \neq 0$ with the choice

$$I = \sqrt{\dot{T}} \qquad J = 0 \qquad K = K_0(1+t^2)^{-1} \tag{4.5}$$

where K_0 is a real constant. The VCNLS equation that corresponds to (4.5) is

$$\mathrm{i}\,u_t + (1+\mathrm{i}\,f_2)\,u_{xx} + (\tilde{g}_1 + \mathrm{i}\,\tilde{g}_2)\,\mathrm{e}^{-[2f_2K_0/(1+t^2)]x^2}\,u\,|u|^2 + \left\{\frac{2K_0(1-\mathrm{i}\,f_2)}{(1+t^2)^2}[2(1+f_2^2)\,K_0 - t]\,x^2\right.$$
$$\left. + \frac{2\mathrm{i}\,K_0(1+f_2^2)}{1+t^2} + \mathrm{i}\,\tilde{h}_2\,T_0\,\mathrm{e}^{-8K_0(1+f_2^2)\tan^{-1}t}\right\}u = 0\,. \tag{4.6}$$

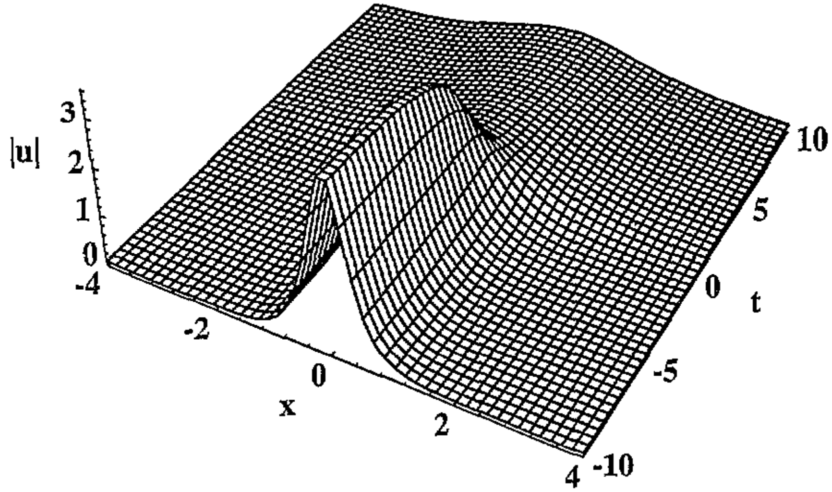

Figure 1. Amplitude $|u(x,t)|$ of the exact solution of example 1.

As one can see, the choice (4.5) does not introduce a time singularity in the VCNLS equation (4.6). Although the model (4.6) is rather complex, it has an exact solution that represents the evolution of a solitary wave into another one. This solution can be obtained from the application of the allowed transformation (4.4) to the following solitary wave solution [28] of the CGL equation:

$$\tilde{u} = \tilde{u}_0 \,[\mathrm{sech}\,(\alpha\tilde{x})]^{1-\mathrm{i}\beta}\, \mathrm{e}^{\mathrm{i}\,\Gamma\tilde{t}} \tag{4.7}$$

where the four parameters $\tilde{u}_0$, α, β and Γ satisfy

$$\begin{aligned}
&\Gamma - \alpha^2(1-\beta^2) + 2\,\tilde{f}_2\,\alpha^2\beta = 0\\
&2\,\alpha^2\beta + \tilde{f}_2\,\alpha^2(1-\beta^2) + \tilde{g}_2 = 0\\
&\alpha^2(\beta^2-2) + 3\,\alpha^2\,\tilde{f}_2\beta + \tilde{g}_1\,\tilde{u}_0^2 = 0\\
&3\,\alpha^2\beta - \tilde{f}_2\,\alpha^2(\beta^2-2) + \tilde{g}_2\,\tilde{u}_0^2 = 0\,.
\end{aligned}$$

Figure 1 illustrates the behaviour of the solution amplitude $|u(x,t)|$ for the choice

$$\tilde{f}_2 = -\tfrac{1}{2} \qquad \tilde{g}_1 = 2 \qquad \tilde{g}_2 = 0 \qquad \tilde{h}_2 = -\tfrac{1}{2} \qquad K_0 = 0.1 \qquad T_0 = 1\,. \tag{4.8}$$

As one can see, the amplitude of the field evolves smoothly between two (different) solitary waves as the time t increases or decreases. This phenomenon is a consequence of the fact that K vanishes as $|t|$ increases. It is interesting to note that although the signs of $\tilde{f}_2$, and $\tilde{h}_2$ in (4.8) represent amplification terms for the CGL equation, the solution of the transformed model (4.6) exhibits a dissipating behaviour as t increases.

Example 2. A particular case of the CGL equation that is of great physical interest is the NLS equation ($\tilde{f}_2 = \tilde{g}_2 = \tilde{h}_2 = \tilde{h}_1 = 0$ and $\tilde{g}_1 =$ constant). The VCNLS equations that can be transformed into the NLS equation can all be considered as completely integrable systems.

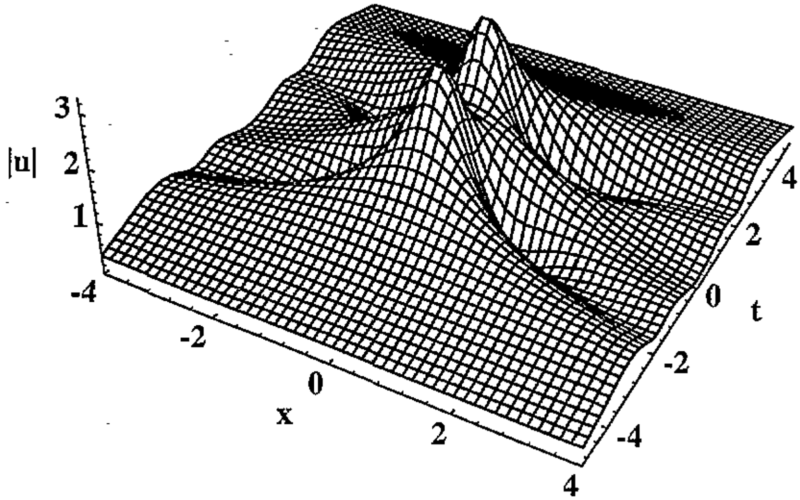

Figure 2. Amplitude $|u(x,t)|$ of the exact solution of example 2 with $K_0 = 1$.

For this example, we make the choice

$$\dot{I}\,I^{-1} = -2 \qquad J = 0 \qquad \dot{K} + 4K^2 = \frac{K_0}{4} \tag{4.9}$$

which yields the following VCNLS equation:

$$\mathrm{i}\,u_t + u_{xx} + \tilde{g}_1\,T_0\,\mathrm{sech}\left(\sqrt{K_0}\,t\right)u\,|u|^2 + \frac{K_0}{4}\,x^2\,u = 0\,. \tag{4.10}$$

In this case again, no time singularity is introduced by the choice (4.9). As an example of the exact solution for (4.10), let us normalize $\tilde{g}_1$ to 2 and apply the allowed transformation (4.4) to the following bound two-soliton solution of the NLS equation:

$$\tilde{u} = \frac{4\,\cosh(3\tilde{x}) + 12\,\cosh(\tilde{x})\,\mathrm{e}^{8\mathrm{i}\tilde{t}}}{\cosh(4\tilde{x}) + 4\cosh(2\tilde{x}) + 3\cos(8\tilde{t})}\,\mathrm{e}^{\mathrm{i}\tilde{t}}\,. \tag{4.11}$$

The amplitude of this solution is symmetric about $\tilde{x} = 0$ and exhibits a periodic peaking in $\tilde{t}$. The allowed transformation turns out to transform the solution (4.11) into a localized structure in the $\{x, t\}$-plane. The amplitude of such a structure is shown in figure 2 for the choice $K_0 = T_0 = 1$. The nonlinear coefficient in (4.10) is partly responsible for the localization as it forces the solution (4.11) to vanishes as $|t|$ increases. The quadratic term in (4.10) also increases the spreading of the structure because of its positive sign. The two humps in figure 2 are the only peaks of solution (4.11) which do not completely disappear for the above choice of parameters K_0 and T_0.

With the choice $K_0 = -1$, the quadratic term in (4.10) with $T_0 = -K_0 = 1$ represents a typical quantum harmonic oscillator potential. However, this choice introduces a singular periodic function 'sec(t)' as nonlinear coefficient. Figure 3 shows the amplitude of an exact time-periodic solution of (4.10) with $T_0 = -K_0 = 1$, which is obtained by applying the allowed transformation (4.4) to the fundamental soliton solution of the NLS equation. The amplitude has a time-periodic singularity on the $x = 0$ axis.

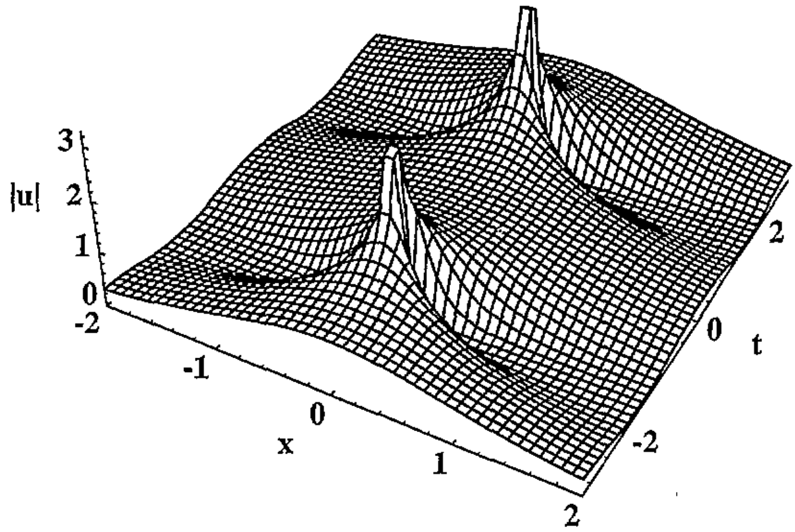

Figure 3. Amplitude $|u(x,t)|$ of the exact solution of example 2 with $K_0 = -1$.

5. Summary and conclusions

The results of this study can be summed up as follows.

(1) Allowed transformations, transforming VCNLS equations of the form (1.1) amongst each other, have the form given in (2.2)–(2.4). If the function $f(x,t)$ is normalized to satisfy $f_1 = 1$, then this normalization is preserved by the allowed transformations of (2.6)–(2.11).

(2) The Lie point symmetry group G of the VCNLS equation has dimension $1 < \dim G \leqslant 5$. The dimension 5 is achieved if and only if the equation is equivalent to one with $f = 1,\ g = \epsilon + \mathrm{i}\, g_2,\ h = 0$ ($\epsilon = \pm 1$, $g_2 =$ constant). The corresponding symmetry algebra is given in (3.30b).

(3) Every VCNLS equation with $\dim G = 4, 3, 2$ can be transformed by an allowed transformation into a representative equation listed in theorems 4, 3, and 2, respectively. The VCNLS equation is always invariant under the one-dimensional (constant) gauge transformations generated by W of (3.13).

(4) It was illustrated in section 4 that VCNLS equations with coefficients depending on x and t in a physically interesting manner can be transformed into equations with constant coefficients. For these many solutions are known and they can, by the inverse transformation be transformed into X- and Y-dependent solutions of the original VCNLS equation.

Further research on this topic is planned in several directions. The first is to study the prototype equations with symmetry groups of dimension 4 and to establish whether any of them are integrable, or in some meaningful sense, 'partially integrable' [52]. The second is to reconsider the derivation of the VCNLS equation in various branches of physics and to analyse the meaning of the individual functions that figure in the equation. The results of this paper can then be used to obtain solutions for physically interesting models. Finally, we plan a systematic study of solutions of VCNLS equations satisfying $\dim G = 5$. We mention that while allowed transformations take solutions of the NLS equation into those of the VCNLS equation, they do not preserve boundary conditions, asymptotic behaviour, etc.

Hence a direct study of Lax pairs, Backlund transformations and other properties of the maximally symmetric VCNLS equations would be of considerable interest.

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